What does the study of Zeno's paradoxes teach us about the nature of the infinite?

In considering this question, it is worthwhile discussing what Zeno thought he was demonstrating when he constructed his well-known paradoxes. As a good Eleatic, Zeno would wish to defend the "one" of Parmenidean monism, and show that both motion and plurality are impossible. Achilles can never catch the tortoise, the journey, in the racecourse paradox, can never be completed, and, in the regressive version, never even started. In the arrow paradox, the arrow is always at rest, and never in flight, and in the paradox of moving blocks, there are also anomalies.

Also, Zeno, as an Eleatic, would wish to claim that he was using pure reason to obtain his results, and not be led down the path of ignorance by relying on sense data. Therefore it is surely wrong to accuse Zeno of merely creating intellectual puzzles to beguile us. He would maintain that the appearance of motion and plurality is just that; an appearance only, an illusion.

Zeno certainly used the infinite in his paradoxes. What would he and other educated Greeks understand by this? For the Greeks, the nearest word to our infinity is απειρον, which means the unlimited, the boundless. However, απειρον also might mean the abyss; nothing. There is more than a whiff of chaos-stuff about απειρον. απερ Is the opposite, meaning bounded. In order for order and justice to prevail, infinity must be bounded. So for the Greeks, there is something of a distaste, even horror of the infinite. This can be seen in the legendary reaction of the Pythagoreans to the discovery that √2 cannot be expressed in terms of ratios of integers. We need to keep this attitude to the infinite in mind when we assess the writings of the early Greek philosophers

The paradoxes pose many important questions regarding the infinite. Even today, when many consider that they have been solved, they are still used in many lectures and text books on mathematics and philosophy. It is not difficult to see why; the paradoxes are set up specifically to be challenged and demolished, but this is not an easy task, and Zeno is no straw man in this respect.

Zeno's spacetime is assumed to be infinitely divisible in the problems associated with the dichotomy and the continuum. It seems intuitively obvious to us that a length or a period of time can be cut into progressively smaller and smaller units, but Zeno showed that such assumptions often led to contradictions. In particular, one contradiction relates to what happens when the separate units are added. Either;

1. The magnitude of the infinite number of units is zero, so they sum to zero, or;
2. The units still have extension, be they ever so small, so that an infinite number of them will sum to an infinite distance, or time.

There is a profound problem of non-commensurability of space and time associated with this paradox.

Perhaps Zeno might have made his own contribution to the debate over the spacetime continuum. In the moving blocks, or stadium problem, Zeno seems to have anticipated the attempt to quantize space and time, where atoms of space – "hodons" and atoms of time "chronons" have been envisaged, and efforts made to analyse their behaviour. The block paradox can arguably be said to challenge the theory that spacetime is "granular" or discrete.

Zeno, through the paradox of the dichotomy also anticipates much of the later discussion of the nature of supertasks. Is it possible for any supertask – a collection of an infinite number of subtasks ever be completed? It seems impossible, since if we keep on subdividing a distance, A – G continually, then we will never reach G.

The problem here is caused by using notions of arithmetic operations which are applicable to integers and rationals, and then expecting them to work in the same unproblematic way when applied to the infinite. Since Zeno, a great deal of work has gone into dealing with the infinite in mathematics and philosophy. Aristotle distinguished actual infinity from potential infinity. Actual infinity exists at a given moment of time, whilst potential infinity is spread over time. Zeno's paradoxes apply to the actual infinite. So passing through a region of space does not mean passing through an actual infinite number of sub regions, but through a potential infinity of them. Ingenious though this idea might be (and it has enjoyed a considerable period of popularity), it suspiciously sounds like an attempt to deny the reality of infinity. It is also useful to consider at this point, some of the early methods of calculation used to measure areas bounded by curves. For example, Archimedes used the idea of an "infinigon" – a polygon with an infinite number of sides – in order to estimate the area of a circle. For any margin of error ε it would be possible to specify an n-sided figure which would be used to assess the circle area within this margin. This is a neat way of sidestepping problems of the infinite, and continued to be used in the early developments of the calculus.

However, Zenos paradoxes still continued to pose problems for mathematicians and philosophers. What they showed was that trying to discuss the notion of infinity with existing language and concepts would always end up producing contradictory results. The problem needed to be approached with a new rigour.
In more recent times, we are now, due to the activities of Cantor and others able to say much more about the nature of infinity, using the language of set theory. Firstly, infinity is more than just an unreachable large (or small) number; it is possible to argue that it is several numbers. A set might consist of a denumerably infinite number of elements. That is to say, the set can be said to be both infinite and countable. The integers and rationals are just such a set. There are also sets which are non-denumerably infinite. The reals are just such a set. We also now know, through the development of the mathematics of limits, that many series, including the one used by Zeno in the dichotomy (1/2, 1/4, 1/8 etc), have finite sums. Also, mathematicians distinguish between *closed* and *open* sets. For example, the closed set of all real numbers in the interval 0 to 1, denoted by [0, 1] includes the numbers 0 and 1. The open set of reals between 0 and 1, denoted by (0, 1) specifically excludes the end points 0 and 1.

Using set theory, limits and open and closed intervals, it is possible to provide a well argued answer to Zeno's paradoxes, and begin to answer the many problems of infinity.

However, some doubts remain. Cantor realized that the set of all sets creates special problems. He had to deal with this problem by denying the existence of such a universal set. A.W Moore (The Infinite, 1990) suggests that this denial of a universal set of sets is very much like the solution to infinity developed by Aristotle; infinity always remains "actual" and never be "potential". Additionally, we might tentatively argue whether mathematics can provide a full definitive answer to our notions of the infinite. We do not occupy dimensionless points in time and space. When I occupy a particular area, I do not occupy a Euclidean point, when I move through space, I do more than just occupy a series of Euclidean points. The mathematical contribution to our knowledge of infinity is invaluable, but it does not completely capture our intuitive understanding of the concept or what is happening during motion. There is still more work to be done.

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